Plasma Astrophysics
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CHAPTER 4

WAVES IN A COLD, COLLISIONLESS PLASMA

In the tenuous plasmas of coronae where fluid turbulence and vortex motions are frequently inhibited by the magnetic field, waves take their place to carry away the energy and momentum of local disturbances. Many kinds of waves are possible in a plasma depending on frequency, species of oscillating particles, restoring force, boundary conditions, inhomogeneity, propagation angle to the magnetic field, etc. In this chapter we shall give an overview of the basic wave modes in homogeneous plasmas ranging from the MHD waves, at periods of the order of minutes under solar conditions, to the high-frequency electromagnetic waves escaping from the corona as radio to X-ray emissions. The common physics of waves as collective phenomena has already been emphasized in Section 3.2 on MHD waves.

What happens in a wave whose frequency exceeds the collision rate? In principle, each particle or group of particles could oscillate in its own way. The velocity distribution may oscillate and not remain Maxwellian. In a first approach that is *not kinetic*, we simply ignore thermal motions in this chapter and replace the velocity distributions by δ-functions. This is what we mean by the adjective 'cold'. The oscillations of the distributions therefore do not play a role here and will be the topic of the next chapter. As in MHD, the cold plasma is considered as a fluid, and the individuality of particles is neglected. The equations of particle and momentum conservation (the moments of Boltzmann's equation) are similar to MHD, except that there are no temperature effects and the different plasma species are not locked to each other by collisions.

4.1. Approximations and Assumptions

Can a plasma be both free of collisions (hot) and have negligible thermal motions (cold)? The adjectives cold, collisionless, non-relativistic, infinite, etc. mean different simplifications in the fundamental equations. Such simplifications are important for understanding the physics in plasma phenomena. The approximations hold as long as the neglected effects are smaller than the phenomenon under scrutiny. Note that there are simplifications that exclude certain effects altogether. The sound wave, for example, does not appear under the cold plasma assumptions, and can only be recovered by allowing for thermal motions. The danger of plasma physics is to exceed the limits of validity set by these simplifying assumptions. In practice, one often does not know these limits. To give an example, interplanetary

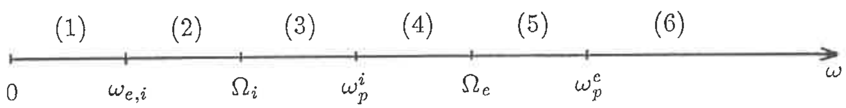

Fig. 4.1. Characteristic frequencies in the order typical for the solar corona. The numbers refer to frequency ranges discussed in the text. $\omega_{e,i}$ is the electron-ion collision frequency (Eq. 2.6.32).

density variations make it questionable to consider the interplanetary medium as an infinite, homogeneous background plasma even for localized processes.

Characteristic frequencies divide a plasma into ranges where different assumptions apply. The parameters of cosmic plasmas vary over many orders of magnitude (see Table 4.1), and this division is not fixed. Depending on the frequency of a wave, the plasma species behave differently. The frequency ranges given as an example in Figure 4.1 for the solar corona have the following properties:

(1) Collisions dominate → MHD waves.

(2) Collisionless plasma, but $\rho^* \approx 0$ → Chew–Goldberger–Low waves (similar to MHD, but the pressure can be different in directions parallel and perpendicular to $\mathbf{B}_0$).

(3) Ions are *unmagnetized*, meaning that the wave oscillates faster than the ion gyration time. The orbit of an ion within a wave period can be approximated by a straight line and is independent of the magnetic field. → Ions and electrons behave differently and have to be described by two fluids.

(4) Ions are practically immobile. → Only electrons are important (one fluid).

(5) Electrons are *unmagnetized*.

(6) Electrons are also immobile. → The plasma increasingly comes to resemble a vacuum.

In every range there is, in principle, a different system of equations suitably describing linear waves. The waves keep their identity, but gradually change their character from one range to the next.

Table 4.1. Typical parameters of various plasmas and their characteristic frequencies.

Plasma	n [cm^{-3}]	T [°K]	B [G]	$\nu_{e,i}$ [Hz]	$\Omega_e/2\pi$ [Hz]	ν_p [Hz]
intergalactic	10^{-6}	10^5	10^{-8}	10^{-12}	$3 \cdot 10^{-2}$	10
interstellar	10^{-2}	10^2	10^{-5}	10^{-3}	30	10^3
interplanetary	10	10^5	10^{-4}	10^{-5}	$3 \cdot 10^2$	$3 \cdot 10^4$
solar corona	10^8	10^6	10	10	$3 \cdot 10^7$	10^8
ionosphere	10^6	10^3	0.1	$3 \cdot 10^3$	$3 \cdot 10^5$	10^7
center of star	10^{24}	10^8	10^6?	10^{14}	$3 \cdot 10^{12}$?	10^{16}
white dwarf	10^{30}	10^8	10^5	*	$3 \cdot 10^{11}$	10^{12}
tokamak	10^{16}	10^8	10^5	10^6	$3 \cdot 10^{11}$	10^{12}

4.2. Cold Plasma Modes

4.2.1. Linearization

A plasma is said to be cold if processes are investigated that do not depend on thermal motion or pressure. The behavior of waves in a cold plasma is derived from the moments of Boltzmann's equation (Eqs. 3.1.16 and 3.1.21) without the MHD approximation and summation over species. These multi-fluid equations – upon neglecting the pressure term ($p = 0$) and collisions ($S^\alpha = 0$) – contain the physics of a cold, collisionless plasma in the frequency ranges 2 – 6 of Figure 4.1. For each species α there is an equation for particle and momentum conservation,

$$\frac{\partial n^\alpha}{\partial t} + \nabla \cdot (n^\alpha \mathbf{V}^\alpha) = 0 \qquad (4.2.1)$$

$$\frac{\partial \mathbf{V}^\alpha}{\partial t} + (\mathbf{V}^\alpha \cdot \nabla)\mathbf{V}^\alpha = \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \frac{1}{c}\mathbf{V}^\alpha \times \mathbf{B}^\alpha) \quad . \qquad (4.2.2)$$

In addition we are using the full set of Maxwell's equations in their classical form,

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \rho^* := \sum_\alpha q_\alpha n_\alpha \qquad (4.2.3)$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho^* \qquad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{J} := \sum_\alpha q_\alpha n_\alpha \mathbf{V}^\alpha \quad . \qquad (4.2.4)$$

The method to extract the linear modes (periodic solutions to small disturbances of an equilibrium) is analogous to the MHD waves (Section 3.2). We study the temporal evolution of the variables of the system having the form $A = A_0 + A_1(\mathbf{x}, t)$, where $A_0 = const$ is the value of the homogeneous, stationary plasma. A_1 is a small deviation compared to the stationary value, satisfying $|A_1| \ll |A_0|$. We shall furthermore assume that $\mathbf{E}_0 = 0$, $\mathbf{J}_0 = 0$, and $\rho_0^* = 0$. The zero-order terms satisfy Equations (4.2.1) – (4.2.4). When the variables of the disturbed plasma are put in, the zero-order terms cancel. Products of first-order variables can be considered of higher order and are neglected. The remaining system of equations is linear,

$$\frac{\partial n_1^\alpha}{\partial t} + \mathbf{V}_0^\alpha \cdot \nabla n_1^\alpha + n_0^\alpha \nabla \cdot \mathbf{V}_1^\alpha = 0 \tag{4.2.5}$$

$$\frac{\partial \mathbf{V}_1^\alpha}{\partial t} + (\mathbf{V}_0^\alpha \cdot \nabla)\mathbf{V}_1^\alpha = \frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \frac{1}{c}\mathbf{V}_1^\alpha \times \mathbf{B}_0 + \frac{1}{c}\mathbf{V}_0^\alpha \times \mathbf{B}_1) \tag{4.2.6}$$

$$\nabla \times \mathbf{E}_1 = -\frac{1}{c}\frac{\partial \mathbf{B}_1}{\partial t} \tag{4.2.7}$$

$$\nabla \cdot \mathbf{E}_1 = 4\pi \rho_1^* \tag{4.2.8}$$

$$\rho_1^* = \sum_\alpha q_\alpha n_1^\alpha \tag{4.2.9}$$

$$\nabla \times \mathbf{B}_1 = \frac{4\pi}{c}\mathbf{J}_1 + \frac{1}{c}\frac{\partial \mathbf{E}_1}{\partial t} \tag{4.2.10}$$

$$\nabla \cdot \mathbf{B}_1 = 0 \tag{4.2.11}$$

$$\mathbf{J}_1 = \sum_\alpha q_\alpha (n_1^\alpha \mathbf{V}_0^\alpha + n_0^\alpha \mathbf{V}_1^\alpha) \quad . \tag{4.2.12}$$

This system of equations is very general and includes a large variety of waves. It also contains the displacement current in Equation (4.2.10) which will be important for high-frequency waves. We shall restrict the discussion to two special cases revealing the most important wave modes and their properties:

(1) $\mathbf{B}_0 \neq 0, \mathbf{V}_0^\alpha = 0$ for all particle species α.

(2) $\mathbf{B}_0 = 0, \mathbf{V}_0^\alpha \neq 0$ with at least two species moving in relation to each other. This case will be studied in Section 4.6.

For a homogeneous stationary background, the Fourier transformation can be carried out by the simple form for plane waves, assuming that all first-order terms have the form

$$A_1(\mathbf{x}, t) \ = \ \bar{A}_1 \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \quad . \tag{4.2.13}$$

All derivatives then become factors,

$$\frac{\partial}{\partial t} \to -i\omega, \ \nabla \to i\mathbf{k} \quad .$$

The dispersion relation can be calculated from Faraday's law (Eq. 4.2.7) if the magnetic field, $\mathbf{B}_1$, is expressed by the electric field. For this purpose we first have to calculate the electric conductivity and the dielectric tensor, and then use Ampère's equation (4.2.10).

4.2.2. Ohm's Law

Electric conductivity is the relation between electric field and current density. Thus we search for an equation of the form of Ohm's law. It follows from the linearized momentum equation (4.2.6) and the assumptions $\mathbf{B}_0 \neq 0$ and $\mathbf{V}_0^\alpha = 0$ that

$$-i\omega \mathbf{V}_1^\alpha = \frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \frac{1}{c}\mathbf{V}_1^\alpha \times \mathbf{B}_0) \quad . \tag{4.2.14}$$

Using the gyrofrequency in vector form, $\mathbf{\Omega}^\alpha := q_\alpha \mathbf{B}_0/(m_\alpha c)$ (Eq. 2.1.4), this transforms into

$$-i\omega \mathbf{V}_1^\alpha + \mathbf{\Omega}^\alpha \times \mathbf{V}_1^\alpha = \frac{q_\alpha}{m_\alpha}\mathbf{E}_1 \quad . \tag{4.2.15}$$

This linear, homogeneous system of three equations can easily be inverted. Let us use a frame of reference in which the z-axis coincides with the direction of $\mathbf{B}_0$. The gyrofrequency vector, $\mathbf{\Omega}^\alpha$, has a non-vanishing component in the z-direction only. We shall use the z-component of the gyrofrequency defined by $\Omega_z^\alpha := q/|q| \ \Omega_\alpha$. The value of Ω_z^α is positive or negative depending on the sign of the charge. Then

$$\mathbf{V}_1^\alpha = \frac{q_\alpha}{m_\alpha}\begin{pmatrix} \frac{i\omega}{\omega^2 - (\Omega_z^\alpha)^2} & \frac{-\Omega_z^\alpha}{\omega^2 - (\Omega_z^\alpha)^2} & 0 \\ \frac{\Omega_z^\alpha}{\omega^2 - (\Omega_z^\alpha)^2} & \frac{i\omega}{\omega^2 - (\Omega_z^\alpha)^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix} * \mathbf{E}_1 \quad , \tag{4.2.16}$$

$$\mathbf{V}_1^\alpha = \widehat{\mathbf{M}}_\alpha * \mathbf{E}_1 \quad . \tag{4.2.17}$$

Equations (4.2.16) and (4.2.17) define a mobility tensor, $\widehat{\mathbf{M}}_\alpha$, describing the effect of the wave electric field on the mean velocity of particle species α. We immediately get Ohm's law and the conductivity tensor $\hat{\sigma}$,

$$\mathbf{J}_1 = \sum_\alpha q_\alpha n_\alpha \mathbf{V}_1^\alpha = (\sum_\alpha q_\alpha n_\alpha \widehat{\mathbf{M}}_\alpha) * \mathbf{E}_1 =: \hat{\sigma} * \mathbf{E}_1 \quad . \tag{4.2.18}$$

It is remarkable that there is a finite conductivity, even without collisions. Inspecting Equation (4.2.16) we find, however, that the conductivity relates to the wave character of the disturbance, and it becomes infinite in the z-direction for $\omega = 0$ in agreement with our assumption of ideal MHD. The conductivity appearing in waves is the result of particle inertia hindering free mobility in the wave fields.

4.2.3. Dielectric Tensor

As a next step we calculate a formal dielectric tensor. So far we have considered the plasma as a set of particles in empty space, and the dielectric properties of the background medium were those of a vacuum. The physical reason for the appearance of dielectric properties is simply that the electromagnetic field and the current of a wave depend on each other. The mobility of the free charges tends to weaken the electric field and thereby induces a magnetic field. One defines a $\mathbf{D}$ vector in Ampère's equation (4.2.10) through

$$i\mathbf{k} \times \mathbf{B}_1 = \frac{4\pi}{c}\mathbf{J}_1 - \frac{i\omega}{c}\mathbf{E}_1 =: -\frac{i\omega}{c}\mathbf{D}_1 \quad . \tag{4.2.19}$$

Using Ohm's law (Eq. 4.2.18) we eliminate $\mathbf{J}_1$ and put $\mathbf{D}_1 := \hat{\epsilon} * \mathbf{E}_1$, where we have defined the dielectric tensor

$$\hat{\epsilon} := \hat{\mathbf{1}} - \frac{4\pi}{i\omega}\hat{\sigma} = \hat{\mathbf{1}} - \sum_\alpha \frac{4\pi q_\alpha n_\alpha}{i\omega}\widehat{\mathbf{M}}_\alpha \tag{4.2.20}$$

$$= \begin{pmatrix} 1 - \sum_\alpha \frac{(\omega_p^\alpha)^2}{\omega^2 - (\Omega_z^\alpha)^2} & -i\sum_\alpha \frac{\Omega_z^\alpha (\omega_p^\alpha)^2}{\omega(\omega^2 - (\Omega_z^\alpha)^2)} & 0 \\ i\sum_\alpha \frac{\Omega_z^\alpha (\omega_p^\alpha)^2}{\omega(\omega^2 - (\Omega_z^\alpha)^2)} & 1 - \sum_\alpha \frac{(\omega_p^\alpha)^2}{\omega^2 - (\Omega_z^\alpha)^2} & 0 \\ 0 & 0 & 1 - \frac{(\omega_p)^2}{\omega^2} \end{pmatrix} \tag{4.2.21}$$

$$=: \begin{pmatrix} \epsilon_0 & i\epsilon_1 & 0 \\ -i\epsilon_1 & \epsilon_0 & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix} \quad . \tag{4.2.22}$$

We have defined the plasma frequencies for each species as well as for the whole plasma by

$$(\omega_p^\alpha)^2 := \frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha} \quad , \tag{4.2.23}$$

$$\omega_p^2 := \sum_\alpha (\omega_p^\alpha)^2 \quad . \tag{4.2.24}$$

The reason for the appearance of the plasma frequency is obvious; ω_p is proportional to the ratio of the electric force to inertia (mass), which also controls the oscillation investigated in Section 2.5. The dielectric properties are therefore usually dominated by the lightest particles – the electrons. The dielectric tensor

reduces to unity for very high frequencies or effectively infinite inertia and becomes equal to the value in a vacuum.

4.2.4. Dispersion Relation

Let us now Fourier transform the linearized Faraday equation (4.2.7)

$$i\mathbf{k} \times \mathbf{E}_1 = \frac{i\omega}{c}\mathbf{B}_1 \quad . \tag{4.2.25}$$

$\mathbf{B}_1$ can be eliminated using Equation (4.2.19). The result is a system of three linear, homogeneous equations,

$$(k^2\hat{\mathbf{1}} - \mathbf{k} \circ \mathbf{k} - \frac{\omega^2}{c^2}\hat{\epsilon}) * \mathbf{E}_1 = 0 \quad . \tag{4.2.26}$$

The tensor product $\circ$ is defined in Appendix A. Equation (4.2.26) only has a non-trivial solution if the determinant of the system equals zero. We thus require

$$\det[(\frac{ck}{\omega})^2\hat{\mathbf{1}} - (\frac{c}{\omega})^2\mathbf{k} \circ \mathbf{k} - \hat{\epsilon}] = 0 \quad . \tag{4.2.27}$$

It is a general dispersion relation for waves in a cold, stationary plasma. The combination $ck/\omega =: \mathcal{N}$ is called the *refractive index*. It controls the refraction (bending) of propagation in inhomogeneous media.

There are four basic roots for linear plasma modes: two *electromagnetic* and two *electrostatic* modes. The electromagnetic waves are related to the two polarization modes of radiation in a vacuum. The charged particles in a plasma participate in the wave oscillations and modify the character of the waves. The electrostatic modes are related to the different oscillation properties of ions and electrons. We have encountered an electrostatic mode in Section 2.5 as the eigenmode of electron oscillations around essentially inert ions. The mode is named *electron plasma wave*. The other electrostatic mode, called *ion wave*, will appear as lower hybrid wave (Section 4.4.1) and as ion acoustic wave (Section 5.2.6). The name 'electrostatic' expresses the fact that there is no magnetic force involved. In the more realistic case, however, when an external magnetic field is present and the wave is not parallel to this field, these waves in general also have magnetic components.

The four basic wave modes do not exist at all frequencies and at all angles to the magnetic field. Furthermore, they behave differently at different frequencies and angles. For this reason, the same mode may carry different names relating to essential physical properties under different approximations. The name of the wave generally characterizes the basic physical principle, not the mode. As for the MHD waves, the physics is best studied in the limiting cases. For an overview we shall ultimately connect the waves at different angles in Section 4.5 to recover the four basic modes.

4.3. Parallel Waves

For a first orientation we have a look at waves that propagate parallel to the magnetic field. We write $\mathbf{k} = (0, 0, k)$, and the determinant (4.2.27) is easily calculated,

$$\epsilon_{\parallel}[(\frac{c^2k^2}{\omega^2} - \epsilon_0)^2 - \epsilon_1^2] = 0 \quad . \tag{4.3.1}$$

Equation (4.3.1) has three solutions: one electrostatic and two electromagnetic. The parallel electrostatic ion wave does not appear in cold plasma and will be discussed in Section 5.2.6.

4.3.1. ELECTROSTATIC WAVES

A first solution to Equation (4.3.1) is readily extracted by putting

$$\epsilon_{\parallel} = 0 \quad . \tag{4.3.2}$$

According to the definition of $\epsilon_{\parallel}$ in Equation (4.2.22),

$$\omega = \omega_p \quad . \tag{4.3.3}$$

We shall show that we have recovered the plasma eigenmode of Section 2.5, except that now all plasma species are allowed to oscillate freely. Thus, the plasma frequency in (4.3.3) is the root mean square over the plasma frequencies of all species. Nevertheless, we shall refer to these waves as *electron plasma waves*, since the oscillation energy of the electrons exceeds that of the ions by the mass ratio $m_i/(m_e Z_i^2)$ (Exercise 4.1). Equation (4.2.26) can only be satisfied (for $\mathbf{E}_1 \neq 0$) if $\mathbf{E}_1$ is parallel to $\mathbf{B}_0$. The mobility equation (4.2.16) then requires that $\mathbf{V}_1$ is also parallel. In other words, the particles oscillate in the same direction as the wave vector. Such a wave is called *longitudinal*.

Furthermore, Ampère's equation (4.2.19) becomes

$$i\mathbf{k} \times \mathbf{B}_1 = -\frac{i\omega}{c}\hat{\epsilon} * \mathbf{E}_1 = 0 \quad . \tag{4.3.4}$$

On the other hand, Equation (4.2.11) requires

$$i\mathbf{k} \cdot \mathbf{B}_1 = 0 \quad . \tag{4.3.5}$$

Both can only be satisfied if $\mathbf{B}_1 = 0$. A wave with this property is generally called *electrostatic*. This special case of a parallel propagating electron plasma wave, where no magnetic effects appear – neither from the background plasma nor in the wave – is named the *Langmuir wave* after one of the pioneers of plasma physics in the 1920s.

The dispersion relation (4.3.3) already indicates how different electron plasma waves are from MHD waves derived in Section 3.2. Only one frequency is possible,

but the wavelength is arbitrary. The phase velocity, ω/k, varies with wavelength; a wave with this property is called *dispersive*. The group velocity vanishes as $\partial\omega/\partial k = 0$. A wave packet thus does not propagate, nor does the wave transport energy. It corresponds to the plasma eigenmode introduced in Section 2.5. We note that these extreme properties of electron plasma waves are considerably moderated when we shall allow temperatures $T \neq 0$ in the following chapter.

Electron plasma waves are excellent density diagnostics in astrophysics. Even allowing for thermal effects, their frequency is closely related to the electron density in the source region (Eqs. 4.2.23 and 4.3.3). But how can these non-propagating waves be observed? The conversion of plasma waves into observable electromagnetic waves is discussed in Section 6.3. Such *plasma wave emission* is the generally accepted process for solar type III radio bursts. An alternative way to make electron plasma waves observable is their investigation by radar, which is practicable in the Earth's magnetosphere.

4.3.2. ELECTROMAGNETIC WAVES

Two more solutions of Equation (4.3.1) can be found by putting the term in brackets equal to zero. First, let

$$\frac{c^2k^2}{\omega^2} = \epsilon_0 + \epsilon_1 := \epsilon_L \quad . \tag{4.3.6}$$

The left side of Equation (4.3.6) is the square of the refractive index, and the equation, combined with the definitions (4.2.22) of ϵ_0 and ϵ_1, is the dispersion relation of a further mode supported by a cold, collisionless plasma. To determine the polarization of this wave, we put the dispersion relation (4.3.6) into Equation (4.2.26) and get

$$\begin{pmatrix} \epsilon_1 & -i\epsilon_1 & 0 \\ i\epsilon_1 & \epsilon_1 & 0 \\ 0 & 0 & -\epsilon_{\parallel} \end{pmatrix} * \mathbf{E}_1 = 0 \quad . \tag{4.3.7}$$

This requires

$$E_{1x} - iE_{1y} = 0 \quad , \tag{4.3.8}$$

$$E_{1z} = 0 \quad . \tag{4.3.9}$$

Equation (4.3.9) states that the wave is *transverse*, meaning that $\mathbf{E}_1$ is perpendicular to $\mathbf{k}$. To comprehend Equation (4.3.8) one has to remember that the first-order variables contain a complex exponential, $\exp[i(k_z z - \omega t)]$, but only the real part is observable. It is straightforward to show that the wave is *left circularly polarized*. (Left here means that the $\mathbf{E}_1$ vector for an observer at a given location rotates counterclockwise when looking along the vector $\mathbf{B}_0$.) Note that neither the sense of rotation (defined by $\mathbf{B_0}$) nor any other property changes for a wave with negative k propagating in the negative z-direction. We may in general call the wave defined by Equation (4.3.6) an *L-wave.*

It follows immediately from the Faraday equation (4.2.7) that $\mathbf{B}_1$ is perpendicular to both $\mathbf{k}$ and $\mathbf{E}_1$. The L-mode thus is *electromagnetic.* It differs from the well-known electromagnetic wave in a vacuum by its inclusion of a current ($\mathbf{J}_1 \perp \mathbf{k}$). This has important consequences if a wave is near resonance to one of the characteristic frequencies of the plasma.

The third solution is found in the analogous way,

$$\frac{c^2k^2}{\omega^2} = \epsilon_0 - \epsilon_1 := \epsilon_R \quad , \tag{4.3.10}$$

being the dispersion relation of the *R-wave.* Again one gets the polarization from Equation (4.2.26). The E_{1z} component vanishes and

$$E_{1x} + iE_{1y} = 0 \quad . \tag{4.3.11}$$

The wave is transverse and *right circularly polarized.* It is not just the mirror image of the L-wave, but differs from it by having other resonances. This we are now going to investigate.

4.3.3. Dispersion Relations of the L and R Waves

L-waves and R-waves are the realizations of the two electromagnetic modes for cold plasma and parallel propagation. They are circularly polarized and can resonate with gyrating particles. In this section we study the effects of resonances on the waves from their dispersion relations. For simplicity, we assume only one species of ions. One derives easily

$$\epsilon_{\substack{L\\R}} = \epsilon_0 \pm \epsilon_1 = 1 - \frac{(\omega_p/\omega)^2}{(1 \mp \frac{\Omega_i}{\omega})(1 \pm \frac{\Omega_e}{\omega})} \quad , \tag{4.3.12}$$

using the modulus of the gyrofrequencies ($\Omega_i := |\mathbf{\Omega}_i|$ and $\Omega_e := |\mathbf{\Omega}_e|$). The upper sign stands for the L-mode, the lower sign for the R-mode. In the following we study this dispersion relation for waves at different frequencies.

- For $\omega \ll \omega_p, \Omega_i, \Omega_e$, both L and R waves have

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{\omega_p^2}{\Omega_i\Omega_e} = 1 + \frac{c^2}{c_A^2} \quad . \tag{4.3.13}$$

This is the dispersion relation for *Alfvén waves* (Eq. 3.2.13),

$$\omega^2 = \frac{k^2c_A^2}{1 + (c_A/c)^2} \quad , \tag{4.3.14}$$

with an additional term stemming from the displacement current, being neglected in MHD. In Section 3.2 we have assumed the form of a linearly polarized Alfvén wave, which can be done in two independent perpendicular directions. Since L and R waves have the same dispersion relations, they can be superposed to a linearly polarized Alfvén wave. At low frequencies, L and R waves are thus identical to Alfvén waves.

- For $\omega \gg \omega_p, \Omega_i, \Omega_e$, one finds from the definitions (4.2.22) and (4.3.12)

$$\epsilon_L = \epsilon_R = 1 \quad . \tag{4.3.15}$$

The dispersion relation then follows immediately as

$$\omega^2 = k^2c^2 \quad . \tag{4.3.16}$$

The equation is identical to the dispersion relation of electromagnetic waves in a vacuum; so the waves are the same as ordinary radiation.

4.3.4. Resonances at the Gyrofrequencies

It is not surprising that the dispersion relation of the L-wave (Eq. 4.3.12) has a singularity at Ω_i, the gyrofrequency of the left circling ions. At this frequency the ions rotate in phase with the wave, feel a constant electric field $\mathbf{E_1}$, and quickly exchange energy with the wave. L-waves below Ω_i are called *ion cyclotron waves.* If propagating into a region where it is in resonance, a wave can be reflected or absorbed, depending on the damping processes. We note that an L-wave propagating into the negative z-direction is also in resonance with the ions.

The analogous process occurs for R-waves at Ω_e. R-waves between the electron and ion gyrofrequencies have peculiar properties deserving special attention. In the range

$$\Omega_i \ll \omega < \Omega_e \ll \omega_p \quad , \tag{4.3.17}$$

Equations (4.3.10) and (4.3.12) can be approximated by

$$\frac{k^2c^2}{\omega^2} \approx \frac{\omega_p^2}{\omega(\Omega_e - \omega)} \quad . \tag{4.3.18}$$

In the form of phase velocity, the dispersion relation (4.3.18) becomes

$$v_{ph} = \frac{\omega}{k} \approx \frac{c\Omega_e}{\omega_p}\sqrt{\frac{\omega}{\Omega_e}(1 - \frac{\omega}{\Omega_e})} = c_A\sqrt{\frac{m_i}{m_e}}\sqrt{\frac{\omega}{\Omega_e}(1 - \frac{\omega}{\Omega_e})} \quad . \tag{4.3.19}$$

These R-waves are called *whistlers,* since their group velocity depends on frequency,

$$v_{gr} = \frac{\partial\omega}{\partial k} \approx 2c_A\sqrt{\frac{m_i\omega}{m_e\Omega_e}}\left(1 - \frac{\omega}{\Omega_e}\right)^{3/2} = 2v_{ph}(1 - \frac{\omega}{\Omega_e}) \quad . \tag{4.3.20}$$

When pulses of whistler waves are excited, the high-frequency waves will arrive first. This property gives the waves their peculiar name. With ordinary long-wave radio receivers one occasionally hears a whistling sound at a decreasing pitch originating from whistler waves excited by terrestrial lightning and propagated through the magnetosphere.

4.3.5. CUTOFFS NEAR ω_p

It is important to discuss carefully the behavior of observable waves near the plasma frequency. For $\omega \gg \Omega_i$ one can approximate Equation (4.3.12) by

$$\left(\mathcal{N}_{\substack{L\\R}}\right)^2 := \epsilon_{\substack{L\\R}} \approx 1 - \frac{(\omega_p/\omega)^2}{1 \pm \Omega_e/\omega} \quad . \qquad (4.3.21)$$

The right side – being equal to the square of the refractive index $\mathcal{N}$ – must be positive for the modes to exist. This condition creates a *cutoff* in frequency below which the waves are evanescent (they do not propagate since their refractive index is purely imaginary). The refractive index equals zero at the cutoff. A propagating wave that meets a cutoff is usually reflected. For R-waves (minus sign in Eq. 4.3.21) the cutoff is at

$$\omega_x \;\approx\; \sqrt{\omega_p^2 + \frac{1}{4}\Omega_e^2} \;+\; \frac{1}{2}\Omega_e \quad . \qquad (4.3.22)$$

For $\omega_p^2 \gg \Omega_e^2$, the cutoff frequency is about $\omega_p + \frac{1}{2}\Omega_e$. In the other extreme, for $\omega_p^2 \ll \Omega_e^2$, the cutoff frequency $\omega_x \approx \Omega_e(1 + \omega_p^2/\Omega_e^2)$. It is easily shown that for R-waves the cutoff is always above $\max(\omega_p, \Omega_e)$, above the two elementary frequencies. The cutoff determines the lowest frequency of waves that can escape from a stellar atmosphere and that can be observed (see Fig. 4.2).

According to Equation (4.3.21) L-waves have a cutoff at $\omega_z = (\omega_p^2 + \frac{1}{4}\Omega_e^2)^{1/2} - \frac{1}{2}\Omega_e$. However, this is a singularity of strictly parallel propagation and is of no practical importance. For all other propagation angles there is a resonance as well as a cutoff at a higher frequency (as will become clear in Section 4.4), namely the plasma frequency. Thus in realistic circumstances, L-waves escape for

$$\omega \;>\; \omega_p \quad . \qquad (4.3.23)$$

The local plasma frequency is therefore the lowest frequency of electromagnetic radiation that can leave a corona from a given height.

The L and R waves are evanescent between the resonance at Ω_i and Ω_e, and their cutoffs at ω_p and ω_x, respectively. This part of the spectrum is called the *stop* region.

Cutoffs and resonances are important for the understanding of electromagnetic radiation from stars. Broadband emission processes cannot excite waves in the stop region between resonance and cutoff in the source. Secondly, a propagating wave may be absorbed. While propagating in an inhomogeneous region, its

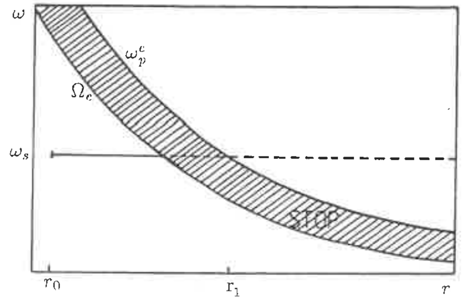

Fig. 4.2. Schematic drawing of whistler wave propagation in a corona. Plasma frequency and electron gyrofrequency vs. radial distance r from the center of the star. No electromagnetic waves can propagate in the 'stop' region.

frequency remains constant. It may reach a stop region if the local characteristic frequencies decrease along the path. Whistler waves, as an example, cannot leave a corona, since their frequency at some higher altitude will exceed the local electron gyrofrequency. In Figure 4.2 a whistler of frequency ω_s originates at r_0. It propagates only until ω_s is about the local gyrofrequency Ω_e, where it is absorbed and/or reflected. Waves observable at ω_s from the outside must originate at $r > r_1$. In extreme cases, tunneling through the stop region is possible. It has been observed in regions with a steep density gradient in the Earth's ionosphere.

4.4. Perpendicular Propagation

4.4.1. ELECTROSTATIC WAVES

The electrostatic modes can easily be extracted from Equation (4.2.26) by scalar multiplication by $\mathbf{k}$ from the left, and the result is

$$\frac{\omega^2}{c^2}\mathbf{k} \cdot (\hat{\epsilon} * \mathbf{E}_1) = 0 \quad . \qquad (4.4.1)$$

Electrostatic waves are longitudinal, i.e. $\mathbf{E}_1 \parallel \mathbf{k}$, as required by the Faraday equation (4.2.25) and $\mathbf{B}_1 = 0$. Equation (4.4.1) leads immediately to the general dispersion relation for longitudinal, electrostatic waves,

$$\mathbf{k} \cdot (\hat{\epsilon} * \mathbf{k}) = 0 \quad . \qquad (4.4.2)$$

It includes the parallel case (Eq. 4.3.2). In the perpendicular case, $\mathbf{k} = (k, 0, 0)$, Equation (4.4.2) gives

$$\epsilon_0 = 0 \quad . \qquad (4.4.3)$$

The definition of ϵ_0, Equation (4.2.22), then yields for $\omega \gtrsim \omega_p \gg \Omega_e$

$$\omega^2 \approx \omega_p^2 + \Omega_e^2 \tag{4.4.4}$$

This mode is called an *upper hybrid wave*, since it is a combination of space charge oscillations with the gyration of electrons. Its frequency, given by Equation (4.4.4), is the upper hybrid frequency. The wave is the analog of the parallel electron plasma waves and may be a cause of radio emission, as will be discussed in Chapters 8 – 10. In the general case with angle θ between $\mathbf{k}$ and B_0, Equation (4.4.4) is replaced by

$$\omega^2 \approx \omega_p^2 + \Omega_e^2 \sin^2\theta \tag{4.4.5}$$

(Exercise 4.2). It also includes the parallel case ($\epsilon_\parallel = 0$), the electron plasma wave of the previous section.

The conditions $\Omega_i \ll \omega \ll \Omega_e$ lead to a completely different perpendicular wave. We find

$$\epsilon_0 \approx 1 + \frac{(\omega_p^e)^2}{\Omega_e^2}\left(1 - \frac{\Omega_e \Omega_i}{\omega^2}\right) = 0 \quad , \tag{4.4.6}$$

and by simple manipulation

$$\omega^2 \approx \frac{(\omega_p^i)^2}{1 + (\omega_p^e/\Omega_e)^2} \quad . \tag{4.4.7}$$

This electrostatic wave is known as the *lower hybrid wave*. Its frequency is called lower hybrid frequency, amounting to $\omega^2 \approx \Omega_i \Omega_e$ for $\omega_p \gg \Omega_e$.

For $\omega_p \ll \Omega_e$ Equation (4.4.7) yields a wave frequency of about ω_p^i. Why does the plasma frequency of the ions appear? This is an interesting piece of wave physics. The wave frequency is small compared to the gyrofrequency of electrons; thus they circle many times per wave period. Electrons therefore remain closely attached to their magnetic field line. The ions, however, need much longer to gyrate than a wave period and move a practically linear orbit during this time. As a result, they appear to be not bound to the magnetic field and can freely move within a wave period. The lower hybrid wave is an oscillation of space charge of the ions. This is in contrast to electron plasma waves, where electrons oscillate around the inert ions (Section 4.3.1). Electrons and ions have changed their roles! For example, lower hybrid waves can be excited by perpendicular ion currents (Chapter 9) and can accelerate electrons parallel to the magnetic field. They are a manifestation of the second electrostatic mode, the ion plasma waves.

4.4.2. Electromagnetic Waves

One may expect that the general dispersion relation, Equation (4.2.27), of waves in a cold, collisionless plasma has the same number of solutions in the cases of parallel and perpendicular propagation. Thus we search for the electromagnetic modes putting $\mathbf{k} = (k, 0, 0)$ into Equation (4.2.27) and find two more modes:

(1) *Ordinary mode*

$$\omega^2 = k^2c^2 + \omega_p^2 \tag{4.4.8}$$

The frequency of these waves always exceeds ω_p, where they have a cutoff. They are linearly polarized with $\mathbf{E}_1 \parallel \mathbf{B}_0$ and $\mathbf{B}_1 \perp \mathbf{B}_0$. As $\mathbf{E}_1 \perp \mathbf{k}$, they are electromagnetic. Since the oscillation of the particles in transverse waves is always parallel to $\mathbf{E}_1$, and in this case also to $\mathbf{B}_0$, the magnetic field does not influence the wave. Therefore B_0 does not appear in Equation (4.4.8). The wave and particles behave as in a non-magnetic plasma. This characteristic property has led to the name *ordinary wave* or *o-mode*. For large $\mathbf{k}$-vectors it is mathematically the same branch of solution as the *L-mode* for $\mathbf{k} \parallel \mathbf{B}_0$. A complication arises at small $\mathbf{k}$, which will be discussed in Section 4.5.

(2) *Extraordinary mode*

There is another mode called *extraordinary* or *x-mode*. The dispersion relation for $\omega \gg \Omega_i$ is

$$\frac{k^2c^2}{\omega^2} \cong 1 - \frac{\omega_p^2(1 - \omega_p^2/\omega^2)}{\omega^2 - (\omega_p^2 + \Omega_e^2)} \quad . \tag{4.4.9}$$

The wave is again electromagnetic and linearly polarized, but has $\mathbf{E}_1 \perp \mathbf{B}_0$. As $\mathbf{V}_1 \parallel \mathbf{E}_1$, the particle oscillations (mostly electrons are involved) are perpendicular to the magnetic field. The higher the electron gyrofrequency, the greater the influence of the magnetic field. The wave differs from the non-magnetic mode, thus it is named *extraordinary*. As one may expect, this oscillation corresponds to the *R-mode* at parallel direction. The refractive index vanishes and causes a cutoff at

$$\omega_x = \sqrt{\omega_p^2 + \frac{1}{4}\Omega_e^2} + \frac{1}{2}\Omega_e \quad . \tag{4.4.10}$$

The resonance (singularity of Eq. 4.4.9) is at the upper hybrid frequency being, however, always below the cutoff frequency.

We note that for both electromagnetic modes the frequencies of wave-particle resonance change with propagation angle between $\mathbf{k}$ and $\mathbf{B}_0$. The cutoff frequencies are the same for parallel and perpendicular propagation, since the lowest frequency is at $k \to 0$ in both modes. In the next section we shall connect the two regimes through intermediate angles.

4.5. Oblique Propagation and Overview

The dispersion relations of the high-frequency waves are shown without derivation in Figure 4.3 as surfaces in $(\omega, k_z, k_\perp)$-space. These waves, also known as *magnetoionic modes*, are the modes supported by the electron gas (Appleton–Hartree approximation). Waves due to the motion of ions (such as the lower hybrid mode) have lower frequency and have been omitted. Four surfaces are clearly visible in Figure 4.3. They are called *branches*. The modes we have previously discussed correspond to cuts along the parallel or perpendicular k-axis.

For later use, thermal effects (assuming T=25 000 K) are included. They limit the $\mathbf{k}$-vector to values smaller than the inverse Debye length (or $kR_e \leq \omega_p/\Omega_e$, Chapter 5). Figure 4.3 presents the case of $\omega_p = 3.22\ \Omega_e$.

In the front edge of Figure 4.3 ($k_z = k_\perp = 0$), the three high frequency modes have their cutoffs at about $\omega_p \pm \frac{1}{2}\Omega_e$, and ω_p, respectively, as expected from $\omega_p^2 \gg \Omega_e^2$. At large $\mathbf{k}$, the correspondence between parallel and perpendicular modes is simple: The parallel R and L modes connect to the x and o modes, respectively, at perpendicular propagation angle. The modes remain transverse, changing gradually from circular polarization into elliptic and finally linear polarization. For intermediate $k_\perp$ ($10^{-2} \ll k_\perp R_e \ll 1$) the Langmuir mode changes into the perpendicular, electrostatic upper hybrid wave. It has cyclotron harmonics at even larger $k_\perp$ called Bernstein waves, a kinetic plasma mode to be discussed in Chapter 8 (for further information see Melrose, 1980). For clarity, the Bernstein modes in the second and fourth bands above Ω_e are indicated in Figure 4.3 only for perpendicular propagation.

At small $\mathbf{k}$, only the R-mode and the whistler branch transform simply. Note the remarkable property of the parallel electron plasma oscillations (Langmuir waves) connecting to the perpendicular o-mode at small k through a region of predominantly electrostatic plasma waves! The L-mode at small k, on the other hand, connects to a predominantly electromagnetic wave (called *z-mode*), becoming gradually electrostatic as $k_\perp$ increases. The transverse character of the z-mode is a thermal effect. For this reason the wave has not appeared in our analysis of cold plasma modes. It exists in the range $\omega_z < \omega < \omega_{uh}$ near perpendicular direction.

Only o(L) and x(R) mode waves can escape from an atmosphere. The general dispersion relation of these waves is

$$\left(\mathcal{N}_{x}^{o}\right)^2 := \frac{k^2c^2}{\omega^2} = 1 - \frac{X}{1-\frac{1}{2}Y_T^2(1-X)^{-1} \pm [\frac{1}{4}Y_T^4(1-X)^{-2}+Y_L^2]^{1/2}} \quad , \tag{4.5.1}$$

where

$$X := (\omega_p/\omega)^2 \tag{4.5.2}$$

$$Y := \Omega_e/\omega\ , \quad Y_T := Y\sin\theta\ , \quad Y_L := Y\cos\theta \quad . \tag{4.5.3}$$

The modes discussed so far in this chapter are referred to as *normal*. If a plasma does not comply with our assumptions (being, for example, inhomogeneous or moving), new modes can appear. One such case will be discussed in the following section.

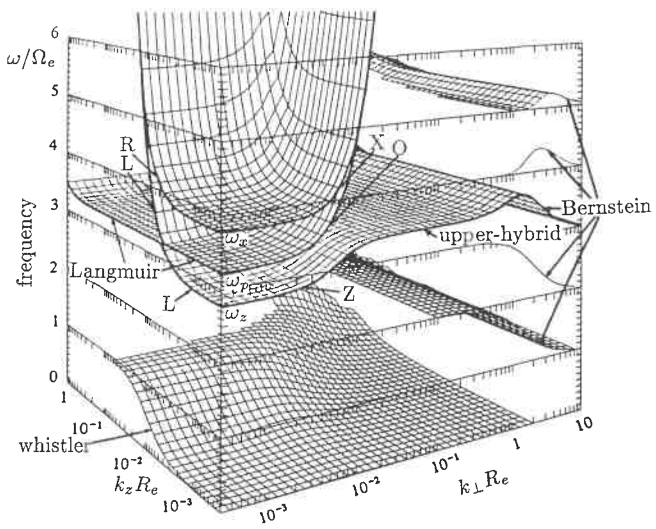

Fig. 4.3. Dispersion surfaces (branches) of high-frequency, collisionless modes in a homogeneous, dense plasma. Regions of strong damping are omitted (after André, 1985).

4.6. Beam Mode

Let us now consider a moving particle species in a plasma that is again cold and collisionless. Moving particle species or beams are ubiquitous in the universe. It is the second case of Section 4.2 with $\mathbf{B}_0 = 0$ and $\mathbf{V}_0^\alpha \neq 0$. We simplify to motions only in the z-direction and consider longitudinal waves. Thus $\mathbf{E}_1 \parallel \mathbf{k} \parallel \mathbf{V}_1^\alpha$. Substituting into Equations (4.2.5) – (4.2.9) yields

$$-i\omega n_1^\alpha + V_0^\alpha ikn_1^\alpha + n_0^\alpha ikV_1^\alpha = 0 \tag{4.6.1}$$

$$-i\omega\mathbf{V}_1^\alpha + \mathbf{V}_0^\alpha ik\mathbf{V}_1^\alpha = \frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \frac{1}{c}\mathbf{V}_0^\alpha \times \mathbf{B}_1) \tag{4.6.2}$$

$$ikE_1 = 4\pi\sum_\alpha q_\alpha n_1^\alpha \quad . \tag{4.6.3}$$

We restrict ourselves to electrostatic waves assuming $B_1 = 0$. The equation of momentum conservation, (4.6.2), yields $\mathbf{V}_1^\alpha$ in relation to $\mathbf{E}_1$, and from the continuity Equation (4.6.1) one extracts n_1^α. Equation (4.6.3) can be rewritten as

$$ikE_1 = 4\pi\sum_\alpha \frac{ikq_\alpha^2 n_0^\alpha}{m_\alpha}\frac{E_1}{(\omega - kV_0^\alpha)^2} \quad , \tag{4.6.4}$$

$$1 - \sum_\alpha \frac{(\omega_p^\alpha)^2}{(\omega - kV_0^\alpha)^2} = 0 \quad . \tag{4.6.5}$$

The dispersion relation (4.6.5) corresponds to the electron plasma wave (Eq. 4.3.3) if one puts $V_0^\alpha = 0$. The Doppler shift, kV_0^α, however produces an oscillation at a frequency lower than ω_p. The waves are referred to as the *beam mode*. They have an important property that we shall illustrate in the following example.

Let us assume that the electrons are in motion, and the ions at rest ($\mathbf{V}_0^i = 0, \mathbf{V}_0^e \neq 0$). Equation (4.6.5) becomes

$$1 = \frac{(\omega_p^i)^2}{\omega^2} + \frac{(\omega_p^e)^2}{(\omega - kV_0^e)^2} =: H(\omega, k) \quad . \tag{4.6.6}$$

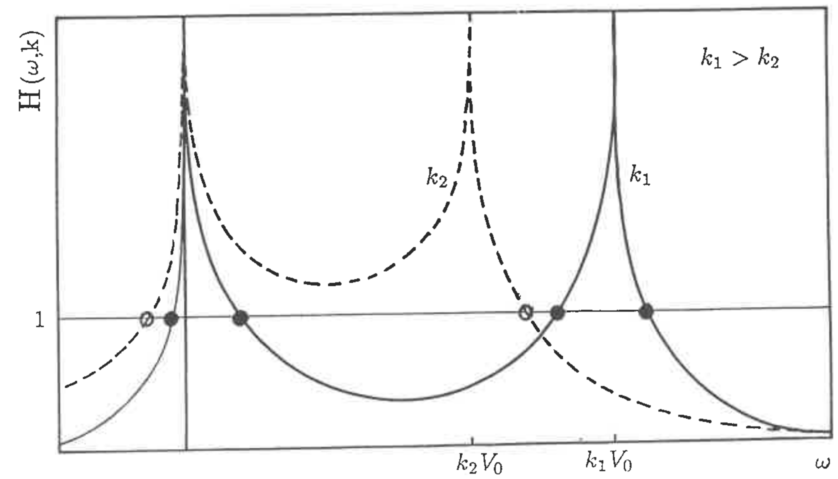

Fig. 4.4. The function $H(\omega, k)$ defined in Equation (4.6.6) vs. ω for two different values of k.

The function $H(\omega, k)$ is shown in Figure 4.4 for two particular waves with different wave vectors. The solution of Equation (4.6.6) are the points of intersection of $H(\omega, k)$ with the horizontal line at $H = 1$. A critical wave vector, k_c, exists below which there are only two real solutions. The two others are complex conjugated,

$$\omega_3 = \omega_r - i\gamma_k \quad , \tag{4.6.7}$$

$$\omega_4 = \omega_r + i\gamma_k \quad . \tag{4.6.8}$$

The solution (4.6.7) is a damped oscillation and is irrelevant. Equation (4.6.8) is of great interest, since it describes a wave with exponentially growing amplitude. Since there is always some small disturbance at the thermal level, this means that the plasma is not stable. The kinetic energy of the moving particles is transformed into wave energy at an increasing rate. Wave amplitude and growth form a feedback cycle; it is an exponential instability. Plasma physics is rich in such phenomena. This particular example is called the *two-stream instability*. The growth of the waves may be extremely fast (Exercise 4.3).

If the growth rate, γ, of Equation (4.6.8) exceeds the damping rate due to collisions in the background plasma, the neglect of collisions is justified. If not, the waves do not grow. The relevant rate is the thermal collision time between the oscillating electrons (test particles) and ions (field particles). It has been evaluated in Equation (2.6.32). The collisional interactions exert a frictional force on the particle motion. The details depend on the excited wave mode and the corresponding fraction of wave energy residing in kinetic motion. As an example, this ratio can be calculated (starting at the equation of motion 4.6.2) to be one half for electron plasma waves at $\omega = \omega_p$. From the work of Comisar (1963), we quote the collisional damping rate of electron plasma waves,

$$\gamma_{\rm coll} = \frac{87 n_e}{T_e^{3/2}} \left(\frac{\ln\Lambda}{20} \right) \quad , \tag{4.6.9}$$

close to the thermal electron-ion collision rate found in Equation (2.6.32). Solar abundances and fully ionized ions have been used to transform n_i into n_e.

The two-stream instability is in fact an extreme case that rarely (if ever) occurs in astrophysical plasmas, since it assumes monoenergetic particles. In the following chapter we shall study the instability in the presence of a finite spread of the velocity distribution. This will take into account that only a fraction of the particles with particular velocities may participate. Such an approach is termed *kinetic* as opposed to the *hydrodynamic* (or 'reactive') instability considered here. As a rule, the instability threshold and growth rate of collisionless waves must be evaluated from a kinetic investigation. Nevertheless, a hydromagnetic treatment, as in this section, may quickly indicate the type of waves to be expected.

Exercises

4.1: Prove that in electron plasma waves the oscillation energy of electrons exceeds that of the ions by the mass ratio $m_i/(Z_i^2 m_e)$ (assuming one ion species only).

4.2: Derive the dispersion relation of electron plasma waves at arbitrary angle α between $\mathbf{k}$ and the magnetic field for $\omega \gg \Omega_e$ (generalization of Eq. 4.4.4).

4.3: Evaluate the dispersion relation of electrostatic waves in cold plasma consisting of two electron beams at velocities $+v$ and $-v$ in opposite directions (neglect ions). At which wave number does the instability grow fastest? Show that the highest growth rate is $\omega_p/2^{3/2}$.

Further Reading and References

Cold plasma waves

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