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CHAPTER 3

# MAGNETOHYDRODYNAMICS

In this chapter we use statistical approaches to coronal plasmas, 'smearing out' individual particles to a fluid. Although their individuality is lost, the particles' collective physical properties are retained. Magnetohydrodynamics (MHD) is a fluid theory. It is appropriate for coronal phenomena that take place on a relatively large scale and are slow. The main branches of MHD are: equilibria, waves, instabilities, and reconnection, on each of which there are already excellent books. These basic processes have been applied to dynamo theory, magneto-convection, flows in the photosphere and chromosphere, coronal loops, prominences, flares, coronal heating, and stellar winds, again on each of which whole books have been written.

This chapter introduces MHD from basic principles and at an elementary level. Its aim is to prepare the ground for the kinetic plasma theory which deals with much smaller-scale and faster processes, and which is the subject of Chapters 5 – 10.

## 3.1. Basic Statistics

### 3.1.1. BOLTZMANN EQUATION

The results of Section 2.6 on collisions in plasmas now allow a deeper understanding of the Boltzmann equation introduced in Chapter 1. If the number of particles of species $\alpha$ is conserved, their distribution function in space and velocity must obey an equation of continuity

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + \ddot{\mathbf{x}} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0 \quad . \tag{3.1.1}$$

This states mathematically that a decrease of the number of particles in an elementary volume of phase space, $f_\alpha(\mathbf{x}, \mathbf{v}, t) d^3x\, d^3v$, is equal to the loss of particles from the volume by particle motion in space and velocity.

We now deal specifically with electromagnetic forces and put

$$\ddot{\mathbf{x}} = \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \quad . \tag{3.1.2}$$

The electric and magnetic fields can be produced collectively or externally, or originate from neighboring particles. The former two are the fields an observer would measure at low spatial or temporal resolution, and they form the macroscopic parts $\mathbf{E}_m$ and $\mathbf{B}_m$. The latter are caused by particles within a Debye radius that generate microscopic fields $\Delta\mathbf{E}$ and $\Delta\mathbf{B}$ fluctuating rapidly in space and time. Thus

$$\mathbf{E} = \mathbf{E}_m + \Delta\mathbf{E} \quad , \tag{3.1.3}$$

$$\mathbf{B} = \mathbf{B}_m + \Delta\mathbf{B} \quad . \tag{3.1.4}$$

We rewrite Equation (3.1.2), dropping the subscripts $m$ and $\alpha$ wherever no confusion is possible,

$$\ddot{\mathbf{x}} = \frac{q}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) + \frac{\Delta\mathbf{F}}{m} \quad , \tag{3.1.5}$$

summarizing the microscopic forces in $\Delta\mathbf{F}$. Equation (3.1.1) becomes the Boltzmann equation (1.4.11),

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = -\frac{\Delta\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} =: \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \quad . \tag{3.1.6}$$

It contains a subtlety: As the left side contains only variables averaged over the ensemble, the right side must also be evaluated to express the average effect of collisional encounters (as was done in Section 2.6).

Equation (3.1.6) is the basis for the physics of fluids as well as kinetic plasma physics. In the fluid approach, the Boltzmann equation is integrated in velocity space. This is the topic of the following two sections. In Chapter 4 we shall come back to Equation (3.1.6), but neglect the collision term. The reduced, collisionless equation is generally referred to as the Vlasov equation. The choice of approach is mainly a question of the time scale or wave frequency.

### 3.1.2. Velocity Moments of the Boltzmann Equation

In the fluid description of a gas the information on the particle velocity distribution is relinquished and replaced by values averaged over velocity space. This is obviously reasonable if the velocity distribution contains little information; in particular, if it is close to Maxwellian and remains so during the course of the process. Let us define such an average for a general variable $A(v)$ with

$$< A(v) > := \frac{\int A(v) f d^3v}{\int f d^3v} \quad . \tag{3.1.7}$$

Using this notation, we define for one species (with mass $m$):

| | | |
|---|---|---|
| particle density | $n := \int f d^3v$ | (3.1.8) |
| average velocity | $\mathbf{V} :=< \mathbf{v} >$ | (3.1.9) |
| stress tensor | $\hat{\mathbf{P}} := nm < \mathbf{v} \circ \mathbf{v} >$ | (3.1.10) |
| pressure tensor | $\hat{\mathbf{p}} := nm < (\mathbf{v} - \mathbf{V}) \circ (\mathbf{v} - \mathbf{V}) >$ | (3.1.11) |
| mean thermal velocity in $i$-direction | $v_t^i := \sqrt{< (\mathbf{v}_i - \mathbf{V}_i)^2 >}$ | (3.1.12) |
| mean thermal energy density | $\mathcal{E} := \frac{1}{2}mn < (\mathbf{v} - \mathbf{V})^2 >=: \frac{3}{2}nk_BT$ | (3.1.13) |



The definitions of stress and pressure make use of the fact that a coronal plasma is very close to an *ideal gas*. Equation (3.1.13) also defines a temperature $T$ corresponding to Equation (1.4.10).

A. *Conservation of Particles*

Now integrate the Boltzmann equation (3.1.6) in velocity space:

$$\int \frac{\partial f}{\partial t} d^3v + \int \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} d^3v + \frac{q}{m}\int (\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} d^3v = \int \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} d^3v \quad . \tag{3.1.14}$$

The third term on the left side is a scalar product, thus a sum of three terms. Each can be integrated by parts and then vanishes. The collision term represents the change of the density by collisions, and is zero due to particle conservation,

$$\int \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} d^3v = \left(\frac{\partial n}{\partial t}\right)_{\text{coll}} = 0 \quad . \tag{3.1.15}$$

Thus the remaining terms are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot(n\mathbf{V}) = 0 \quad . \tag{3.1.16}$$

The integration of the Boltzmann equation (3.1.16) has yielded the *equation of continuity* of particle density – a result of particle conservation.

B. *Conservation of Momentum*

Let us multiply the Boltzmann equation by $mv_k$, where $k$ denotes here one of the three velocity coordinates, and integrate over all velocity space. The first term then becomes

$$m\frac{\partial}{\partial t}\int v_k \cdot f d^3v = \frac{\partial}{\partial t}\cdot(mnV_k) \quad , \tag{3.1.17}$$

and corresponds to the temporal change of momentum density. The second term is the force per unit volume due to a pressure gradient,

$$m\int v_k\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} d^3v = \frac{\partial}{\partial \mathbf{x}}\cdot(nm < v_k\mathbf{v} >) = \sum_i \frac{\partial P_{ki}}{\partial x_i} \quad . \tag{3.1.18}$$

The subscripts $k$ and $i$ refer to tensor and vector elements. Replacing the electromagnetic force, $q(\mathbf{E} + (\mathbf{v} \times \mathbf{B})/c)$, and the other forces by the general symbol $\mathbf{F}$, we find for the third term

$$\int v_k \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{v}} d^3v = - \int f \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{F} v_k) d^3v = -n < F_k > \tag{3.1.19}$$

and for the collision term

$$m \int v_k \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} d^3v = (\frac{\partial}{\partial t} nmV_k)_{\text{coll}} =: S_k \quad , \tag{3.1.20}$$

representing the change of momentum by collisions. Combining Equations (3.1.17) – (3.1.20), the first moment of the Boltzmann equation becomes

$$\frac{\partial}{\partial t}(nmV_k) + \sum_i \frac{\partial P_{ki}}{\partial x_i} = n < F_k > + S_k \quad . \tag{3.1.21}$$

Equation (3.1.21) expresses the conservation of momentum and is usually referred to as the *equation of motion.* These equations hold for each particle species. The different species are coupled by the collision term, $\mathbf{S}$, of each species.

As an immediate application of the above derivations, Exercise 3.1 considers the electric field that arises from the smaller gravitational force on the electrons compared to ions. In a stationary equilibrium this electric field reduces the weight of the protons by half, and instead the electrons are pressed into the star. The extent of electron and proton atmospheres and their scale heights become equal. Compared to the Lorentz force of coronal magnetic fields, the effect of the electric field on individual particle orbits is usually small.

C. *Conservation of Energy*

Analogously, the second moment yields energy conservation,

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (n < \frac{1}{2} mv^2 \mathbf{v} >) = n < \mathbf{F} \cdot \mathbf{v} > + \Pi \quad . \tag{3.1.22}$$

The right side of Equation (3.1.22) consists of two terms. The first represents the work done by the force $\mathbf{F}$. It may include, for example, acceleration by an electric field, emission of radiation, or a heat input. $\Pi$ is the change in energy density due to collisions.

The second term on the left side of Equation (3.1.22), the energy flux, enters as a new variable, just as particle and momentum flux have appeared in the equations of particle and momentum conservation, respectively. The term includes changes in flow velocity, as well as changes in thermal conduction, $-\nabla \cdot (\hat{\kappa} * \nabla T)$. Classical thermal conduction is controlled by particle collision. The thermal conductivity tensor, $\hat{\kappa}$, is diagonal. Parallel to the magnetic field, thermal conduction is primarily by the more mobile electrons. Across the magnetic field, ions – having a larger gyroradius – are primarily responsible.

The thermal conductivity parallel to the magnetic field is proportional to the density multiplied by the thermal velocity and the mean free path ($l_{\text{mfp}} = v_{te} t_t^e$). In equilibrium the diffusion of electrons builds up an electric field such that the current is cancelled. This electric field reduces the heat flow by a factor $a \approx 0.5$. For solar abundances one finds from Spitzer (1962)



$$\kappa_z \approx a\ n_e v_{te}^2 t_t^e k_B \approx 1.72 \cdot 10^{-5} \frac{T_e^{5/2}}{\ln\Lambda} \quad [\text{erg s}^{-1}\ \text{K}^{-1}\ \text{cm}^{-1}] \ . \tag{3.1.23}$$

For thermal conduction perpendicular to the magnetic field, we replace the thermal velocity in Equation (3.1.23) by the ion gyroradius $R_i$ divided by the thermal collision time of ions, $t_t^i$, namely

$$\kappa_\perp \approx \frac{a n_i k_B R_i^2}{t_t^i} \approx 8.9 \cdot 10^{-13} \frac{n_e^2 (\ln \Lambda)^2}{T^3 B^2} \kappa_z \quad [\text{erg s}^{-1}\ \text{K}^{-1}\ \text{cm}^{-1}] \ . \tag{3.1.24}$$

Equation (3.1.24) assumes $\Omega_i t_t^i \gg 1$. Solar abundances have been assumed, $T$ is in degrees kelvin, $B$ in gauss, and $\Lambda$ has been given by Equations (2.6.16) and (2.6.17). Note that Equations (3.1.23) and (3.1.24) assume plasmas having nearly Maxwellian velocity distributions and particles having a mean free path much shorter than the temperature scale length. These approximations become questionable in the transition layer, in flares, and in solar and stellar winds.

### 3.1.3. Elementary Magnetohydrodynamics (MHD)

Some processes in coronal physics are slow, and if a plasma process is slow enough, the physics becomes much simpler. Let us introduce a characteristic time of the process, $t_{\text{char}}$, which may be a travel time, a wave period or the inverse of a growth rate. We shall assume that the process is slow enough for particle collisions to smear out deviations from a Maxwellian velocity distribution, and for differences between particle species in temperature and average velocity to become unimportant. Furthermore, we assume that spatial limitations and boundary effects are unimportant. It may then be advantageous to sum particle number, momentum and energy density over all species $\alpha$, and use the equations of conservation. An important consequence is that the sum over the collision terms of all species vanishes, since the total momentum is conserved,

$$\sum_\alpha \mathbf{S}^\alpha = 0 \quad . \tag{3.1.25}$$

We shall use the following definitions:

| | | |
|---|---|---|
| total mass density | $\rho := \sum_\alpha m_\alpha n_\alpha$ | (3.1.26) |
| total stress tensor | $\hat{\mathbf{P}} := \sum_\alpha \hat{\mathbf{P}}_\alpha$ | (3.1.27) |
| mass flow velocity | $\mathbf{V} := (\sum_\alpha m_\alpha n_\alpha \mathbf{V}_\alpha)/\rho$ | (3.1.28) |
| barycentric pressure | $\hat{\mathbf{p}} := \sum_\alpha m_\alpha n_\alpha < (\mathbf{v} - \mathbf{V}) \circ (\mathbf{v} - \mathbf{V}) >_\alpha$ | (3.1.29) |
| | $= \hat{\mathbf{P}} - \mathbf{V} \circ \mathbf{V} \rho$ | (3.1.30) |

The tensor $\hat{\mathbf{p}}$ is defined in relation to the motion of mass, being dominated by the ions (baryons). Furthermore, one finds from the definitions (1.4.6) and (1.4.7)

charge density $\rho^* = \sum_\alpha q_\alpha n_\alpha$ , (3.1.31)

current density $\mathbf{J} = \sum_\alpha q_\alpha n_\alpha \mathbf{V}_\alpha$ . (3.1.32)

A. *MHD Equations and Approximations*

We now multiply Equation (3.1.16) by $m_\alpha$ and sum over $\alpha$ to get the equation of mass conservation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad . \tag{3.1.33}$$

Summing the equation of momentum conservation (3.1.21) over the particle species yields, for electromagnetic forces and gravity,

$$\frac{\partial}{\partial t}(\rho V_k) + \frac{\partial}{\partial x_i} P_{ki} = \sum_\alpha n_\alpha [q_\alpha (\mathbf{E} + \frac{1}{c}\mathbf{V}_\alpha \times \mathbf{B})_k + m_\alpha g_k] \quad , \tag{3.1.34}$$

which we rewrite in vector form using Equations (3.1.30) and (3.1.33),

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho (\mathbf{V} \cdot \nabla)\mathbf{V} + \nabla * \hat{\mathbf{p}} = \rho^* \mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B} + \rho \mathbf{g} \quad . \tag{3.1.35}$$

The equation states that the change in momentum density (first term on the left side) is caused by the flow of momentum (second term), pressure gradient (third term) and external forces (right side). The equations for mass and momentum conservation, (3.1.33) and (3.1.35), contain the variables $\rho, \mathbf{V}, \hat{\mathbf{p}}, \rho^*, \mathbf{E}, \mathbf{J}, \mathbf{B}$. The combined system of equations including Maxwell's (Eqs. 1.4.1 – 1.4.5) comprises a smaller number of equations and initial conditions than variables. The following approximations are generally used in MHD to close the system:

(1) Charge neutrality, $\rho^* = 0$, is suggested by the Debye shielding effect for large-scale and slow processes. Ampère's equation (1.4.2) then requires $\nabla \cdot \mathbf{J} = 0$.

(2) $\mathbf{J} = \sigma \mathbf{E}'$ in the frame of reference of the system moving with $\mathbf{V}$.

$= \sigma(\mathbf{E} + (\mathbf{V} \times \mathbf{B})/c)$ in the laboratory system.

The electric conductivity $\sigma$ has to be determined from electron–ion collisions (Section 9.2). Often $\sigma = \infty$ is assumed; then $\mathbf{E} = -(\mathbf{V} \times \mathbf{B})/c$. This is called *ideal MHD*.

(3) We assume that the characteristic time of the process exceeds the rate of collisions (Eq. 2.6.31) and the gyroperiods,



$$t_{\rm char} \gg t_{\rm coll}, \frac{2\pi}{\Omega_i}, \frac{2\pi}{\Omega_e} \quad . \tag{3.1.36}$$

Then the distributions of particle velocity, and consequently the pressure, are isotropic. The relation

$$p^\alpha_{ki} = \delta_{ki} p^\alpha = n_\alpha k_B T_\alpha \tag{3.1.37}$$

then follows from Equations (3.1.11) and (3.1.13) where an ideal gas has been assumed. Equation (3.1.37) is called the *equation of state*. The total pressure is the sum of the partial pressures,

$$p = \sum_\alpha p_\alpha \quad . \tag{3.1.38}$$

This is a consequence of Equation (3.1.36), requiring that the collisions have time to equalize the temperatures of all species. Thus,

$$p = \sum_a n_\alpha k_B T = \frac{\rho}{\overline{m}} k_B T \quad . \tag{3.1.39}$$

For a fully ionized plasma with solar abundances, $\sum_\alpha n_\alpha = 1.92 n_e$, and the mean mass is $\overline{m} = 0.60 m_p$.

(4) In place of the energy equation (3.1.22), a process may be approximated by one of the following special equations of state adapted to the problem:

incompressible $\Longleftrightarrow \rho(t) =$ const. (implying $\nabla \cdot \mathbf{V} = 0$, cf. Eq. 3.1.33)

isothermal $\Longleftrightarrow \rho \propto p$

adiabatic $\Longleftrightarrow p\rho^{-5/3} =$ const .

Furthermore, several implicit assumptions are generally made: (*i*) In the evaluations of $\sigma$ and $p$ the plasma is assumed to be close to thermodynamic equilibrium; and (*ii*) the internal structure of the plasma is neglected. This means that the size of the plasma and the phenomena of interest (e.g. the wavelength) are assumed to exceed the ion gyroradius, the Debye length, and the mean free path. If we also neglect relativistic effects and the displacement current, the basic equations of MHD are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{3.1.40}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V} \cdot \nabla)\mathbf{V} + \nabla p = \frac{1}{c}\mathbf{J} \times \mathbf{B} + \rho \mathbf{g} \tag{3.1.41}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \tag{3.1.42}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad (3.1.43)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (3.1.44)$$

$$\mathbf{J} = \sigma(\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B}) \qquad (3.1.45)$$

Energy Equation or Specialized Equation of State .

Instead of the energy equation supplemented by a heat conduction equation, the system of equations may be closed by an appropriately specialized equation of state. MHD in these various forms finds many applications in astrophysics from the liquid metallic core of planets to the flows of stellar winds.

B. *Electric Fields*

The displacement current $1/4\pi\ \partial\mathbf{E}/\partial t$ (Eq. 1.4.2) has been neglected in Equation (3.1.42) for the following reason. Assume a scale length of the fields, $H$, and a characteristic time for the process, $t_0$. Faraday's equation (3.1.43) implies $E \approx -vB/c$, where $v = H/t_0$. Therefore, the displacement current is about $v^2B/(Hc)$, and is much smaller than $c\nabla \times B \approx cB/H$ for $v^2 \ll c^2$.

The electric field, $\mathbf{E}$, can be calculated readily from Ohm's law (3.1.45), and $\mathbf{J}$ follows from Ampère's law (3.1.42). Putting them into Faraday's equation (3.1.43) yields the *induction equation*

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{V}) = \frac{c^2}{4\pi\sigma}\nabla^2\mathbf{B} \quad , \qquad (3.1.46)$$

where we have used the vector identity (A.10) and $\nabla \cdot \mathbf{B} = 0$.

It is important to note that in ideal MHD $\mathbf{B}$ and $\mathbf{V}$ are the fundamental variables. $\mathbf{J}$ and $\mathbf{E}$ are secondary and can be calculated if required. $\mathbf{B}$ and $\mathbf{V}$ are determined by the induction equation and the equation of motion. Once they are found, $\mathbf{J}$ and $\mathbf{E}$ follow. Note in particular that $\mathbf{J}$ is not driven by $\mathbf{E}$ in ideal MHD, and so standard circuit theories are inappropriate.

C. *MHD Properties*

Equations (3.1.40) – (3.1.45) are only the starting point of MHD. They can describe an enormous number of phenomena. Over the years, much practical knowledge on MHD plasma behavior has accumulated, some key elements of which we present here.

• The strong coupling between a magnetic field and matter, a very important property of plasmas, follows immediately from the induction equation (3.1.46). This consequence of Equation (3.1.46) can best be appreciated by integrating it over a plane surface $A$. The left side then expresses the total change in magnetic flux through the surface, $\Phi := \int_A \mathbf{B} \cdot d\mathbf{s}$. Let the boundary $A'$ of $A$ be defined by some floating corks in the fluid. The boundary thus moves with the mass flow velocity defined in Equation (3.1.41). The first term of Equation (3.1.46) is the change of flux due to a variation in $\mathbf{B}$, the second term corresponds to the change caused by the motion of the boundary of $A$. Thus we write the left side of the integrated Equation (3.1.46) as a total derivative,



$$\frac{d\Phi}{dt} := \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_{A'} (\mathbf{B} \times \mathbf{V}) \cdot d\mathbf{l} \ \approx\ \mathrm{sig}(\nabla^2\mathbf{B} \cdot d\mathbf{s})\frac{c^2\Phi}{4\pi H_B^2\sigma} \quad . \qquad (3.1.47)$$

The scale length $H_B^2 = B/\nabla^2 B$ has been introduced. The time

$$\tau = \frac{4\pi H_B^2\sigma}{c^2} \qquad (3.1.48)$$

is needed for diffusion of the magnetic field through the plasma. It is an upper limit of the decay time for magnetic flux concentration. For photospheric conditions ($H_B \approx 10^8$cm, $\sigma \approx 10^{12}$ Hz) the diffusion time is $\tau \approx 4$ years. Obviously, sunspots with typical lifetimes of weeks cannot build up or decay by diffusion of field lines. It will be shown in Section 9.2.1 that the conductivity $\sigma$ is inversely proportional to the collision rate. Equation (3.1.48) states that the magnetic diffusion involves a drag on the particle motion due to collisions. The same drag and finite conductivity also dissipate energy (Ohmic heating). Therefore, the decay of the magnetic field by diffusion is due to dissipation of energy through Ohmic heating and *vice versa*.

If one studies processes shorter than the diffusion time, $\sigma$ may be considered infinite. Equation (3.1.47) then states that the magnetic flux is conserved in a surface moving with the plasma. The *magnetic field is frozen into the matter*, meaning that a fluid element is attached to its field line like a pearl on a string. Density differences can be smoothed out easily along field lines, but not across them, which is why it has become common to think of a field line as a real object, though it is only a mathematical construct. The magnetic field line and the plasma stay together whether the matter is moving and pulling the field along or *vice versa*.

• The induction equation (3.1.46) relates the temporal change of the magnetic field to convection (second term) and diffusion (right side). An MHD process is conveniently characterized by the ratio of these terms, a dimensionless parameter known as the *magnetic Reynolds number*

$$R_m := \frac{4\pi}{c^2}\ V_\perp\ \sigma\ H_B \quad , \qquad (3.1.49)$$

where $V_\perp$ is the velocity component perpendicular to $\mathbf{B}$, and $H_B$ is the scale length of the magnetic field. $R_m$ is infinite for ideal MHD and much larger than unity for most coronal processes.

• It is convenient to eliminate $\mathbf{J}$ in the equation of momentum conservation (3.1.41) to give

$$\rho\frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla(p + \frac{B^2}{8\pi}) = \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi} + \rho\mathbf{g} \quad . \qquad (3.1.50)$$

The right side differs from zero if the field curves or converges and so exerts an anisotropic tension. Equation (3.1.50) contains an important relation between magnetic field and plasma pressure. Let us consider a stationary plasma (i.e. $\mathbf{V} = 0, \partial/\partial t = 0$). The first two terms of Equation (3.1.50) then vanish. The rest of the equation states that $p$ and $B^2/8\pi$, the magnetic energy density, are in equilibrium with tensions produced by magnetic field inhomogeneity. For straight magnetic field lines the right side of Equation (3.1.50) vanishes, and the combined magnetic field lines the right side of Equation (3.1.50) vanishes, and the combined pressures of the plasma and the magnetic field become constant in space. Localized strong field regions then must have lower plasma pressure to balance the outside pressure. As the magnetic pressure in coronae is usually higher than the plasma pressure, large variations of the latter can be accommodated by small variations of the magnetic field. This is the reason for extreme inhomogeneity of coronae.

Consider as an example an idealized 'flux tube', a plasma structure outlined by magnetic field lines. Let its density be one tenth of the outside density; assume the same temperature and no magnetic field outside. The magnetic field of the flux tube must then balance 9/10 of the outside pressure. At the bottom of the solar photosphere, where $p_{\rm out} \approx 2 \cdot 10^5$ dyne cm$^{-2}$, the magnetic field would be of order 2100 G, comparable to the maximum field strength measured in sunspots.

- The ratio of the thermal pressure to the magnetic pressure,

$$\beta := \frac{8\pi p}{B^2} \approx \left(\frac{\omega_p^e}{\Omega_e}\frac{2v_{te}}{c}\right)^2 \quad , \tag{3.1.51}$$

is an important dimensionless parameter known as the *plasma beta*. The second equation uses definitions of Chapter 2 and is accurate for a hydrogen plasma with $T_e = T_i$. Coronae are usually low-beta plasmas ($\beta \ll 1$), the solar wind has a beta of order unity or higher.

- In a stationary atmosphere with uniform or negligible magnetic field, Equation (3.1.50) yields

$$\frac{\partial p}{\partial h} = -\rho g(h) \quad . \tag{3.1.52}$$

The right side is the gravitational force per unit volume. For constant $g$ and in isothermal conditions, Equation (3.1.52) becomes the barometric equation having an exponential solution

$$n(h) = n(h=0)\exp(-h/H_n) \tag{3.1.53}$$

with a density scale height

$$H_n = \frac{p}{\rho g} = 5.00 \cdot 10^3 \frac{T}{g_\odot} \text{ cm} \quad , \tag{3.1.54}$$

where $g_\odot$ is in units of the gravity in the solar photosphere (cf. Appendix C), $T$ is in degrees kelvin, and solar abundances have been used.



## 3.2. MHD Waves

Waves are an important example of collective particle motion. Some or all particles in a volume element are slightly displaced by a local disturbance, but a restoring force – due to a pressure gradient, an electric or magnetic field, gravity, etc. – drives them back to the initial position. They overshoot and oscillate collectively around the equilibrium position. It is not the individual particle that is of primary interest, but rather the collective phenomenon, the average properties of the oscillation, and its propagation as a wave.

Spacecraft in the solar wind plasma have observed a bewildering variety of oscillations. In general, the wiggling particle motions, electric and magnetic field oscillations, and the combination of all three may be exceedingly complex. We shall restrict ourselves to (*i*) small disturbances of (*ii*) homogeneous, unlimited background plasmas. Under these conditions the plasma oscillations occur in only a few basic wave modes. These types of waves have characteristic *polarization* (relations between the wave vector, the vectors of particle motion, and wave fields) and *dispersion relations* (which relate the wave frequency $\omega$ to the wave vector $\mathbf{k}$).

There are two often neglected consequences of the fact that the equations of a plasma are not linear: (*i*) The superposition of two solutions does not necessarily solve the equations any longer; and (*ii*) the possibilities for waves are not exhausted by finding all the periodic and small-amplitude (linear) solutions. Nevertheless, we shall examine small-amplitude waves, as they exhibit the basic physics and form the basis of our understanding of oscillatory plasma phenomena. They are frequently sufficient for the description of waves.

Here we only derive the basic theory of MHD waves. More complete treatments can be found in many specialized textbooks. The derivation of the wave modes in the MHD approximation is valid at the low-frequency end of the spectrum. The method, however, follows a standard pattern for waves in all regimes and will be used similarly at higher frequencies. One first considers a plasma in equilibrium and perturbs it slightly such that the deviations are much smaller than the initial values. The main idea is to approximate the system of equations by a system that is *linear* in the variables of the deviation. The resulting 'linear' disturbance is analyzed to see whether it propagates as a wave having the form $\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$. The goal of the mathematical discussion is to find the dispersion relation and the polarization.

### 3.2.1. Linearization

The MHD equations (3.1.40) – (3.1.45) with the choice of the adiabatic equation of state appropriate for most waves,

$$p\rho^{-5/3} = \text{const} \quad , \tag{3.2.1}$$

can be linearized by

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1(\mathbf{x},t), \quad p = p_0 + p_1(\mathbf{x},t), \quad \mathbf{V} = \mathbf{V}_1(\mathbf{x},t) \quad , \tag{3.2.2}$$

$$\rho = \rho_0 + \rho_1(\mathbf{x}, t) \quad , \tag{3.2.3}$$

if the variables with subscript 1 (disturbance) are much smaller than the stationary, homogeneous variables (subscript 0). These *zero-order* variables cancel out when Equations (3.2.2) and (3.2.3) are inserted into the MHD equations. Let us assume ideal MHD ($\sigma = \infty$) and combine the equations of Faraday and Ohm, (3.1.42) and (3.1.45), to eliminate $\mathbf{E_1}$. Upon neglecting the products of first-order terms, we get a homogeneous system of linear equations,

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{V}_1 = 0 \tag{3.2.4}$$

$$\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} + \nabla(p_1 + \frac{2\mathbf{B}_0 \cdot \mathbf{B}_1}{8\pi}) - \frac{(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1}{4\pi} = 0 \tag{3.2.5}$$

$$\nabla \times (\mathbf{V}_1 \times \mathbf{B}_0) = \frac{\partial \mathbf{B}_1}{\partial t} \tag{3.2.6}$$

$$\frac{p_1}{p_0} = \frac{5}{3}\frac{\rho_1}{\rho_0} \quad . \tag{3.2.7}$$

Counting equations and first-order variables, we find the same number. We now show, as one would expect for linearized waves, that the equations do not determine the amplitude of the disturbance. The temporal derivative of Equation (3.2.5) is

$$\rho_0 \frac{\partial^2 \mathbf{V}_1}{\partial t^2} + \nabla(\frac{\partial p_1}{\partial t} + \frac{\mathbf{B}_0 \cdot \frac{\partial}{\partial t}\mathbf{B}_1}{4\pi}) - \frac{(\mathbf{B}_0 \cdot \nabla)\frac{\partial}{\partial t}\mathbf{B}_1}{4\pi} = 0 \quad , \tag{3.2.8}$$

in which we eliminate all first-order variables except $\mathbf{V}_1$ using Equations (3.2.4), (3.2.6), and (3.2.7),

$$\begin{aligned} \rho_0 \frac{\partial^2 \mathbf{V}_1}{\partial t^2} - p_0 \frac{5}{3} \nabla(\nabla \cdot \mathbf{V}_1) + \frac{1}{4\pi} \nabla[\mathbf{B}_0 \cdot (\nabla \times (\mathbf{V}_1 \times \mathbf{B}_0))] \\ - (\frac{\mathbf{B}_0}{4\pi} \cdot \nabla)[\nabla \times (\mathbf{V}_1 \times \mathbf{B}_0)] = 0 \end{aligned} \quad . \tag{3.2.9}$$

This equation describes the evolution of an arbitrary initial disturbance of the mass flow velocity, $\mathbf{V}_1$, in space and time. The amplitude of $\mathbf{V}_1$ is a constant factor in space and time, and cancels. The other first-order variables can be evaluated in a similar way. For instance, $\mathbf{B}_1$ can be found from Equation (3.2.6).

### 3.2.2. DISPERSION RELATION AND POLARIZATION (PARALLEL PROPAGATION)

Most wave equations like (3.2.9) look very complicated. Usually they can be interpreted by evaluating them for simple cases, like parallel or perpendicular wave propagation to the magnetic field. These simplifications should not be misunderstood as helpless attempts in the face of sheer complexity, nor as likely cases (other propagation angles to the magnetic field may be equally frequent). The extreme cases and approximations identify important physics. Even the names of waves refer to physical properties in simple limits rather than to a particular mode. For this reason a mathematically identical wave (a particular branch of the solution) may have different names at parallel and perpendicular propagation, at low and high frequency. This may be surprising or confusing in the beginning, but it indicates important differences in the physics of the wave.



We shall first solve Equation (3.2.9) for waves propagating parallel to the external magnetic field, $\mathbf{B_0}$, assumed to be in the $z$-direction. For simplicity we take the disturbance as an infinitely extended plane wave with an amplitude independent of space and time. Thus we write

$$\mathbf{V}_1 = \mathbf{V}_1^{\perp} + \mathbf{V}_1^{z} = (\overline{\mathbf{V}}_1^{\perp} + \overline{\mathbf{V}}_1^{z}) e^{i(kz - \omega t)} \tag{3.2.10}$$

The bar indicates that the quantity is an amplitude. It will later be omitted when no ambiguities are possible. The $z$ coordinate in the exponent marks the only spatial variation. It causes the phase of the wave to propagate in the positive $z$-direction if $k > 0$, and in the negative $z$-direction if $k < 0$. We use here the convention that the wave frequency is always positive. (Note that the inverse convention, $\omega < 0$ and $k > 0$ for negative wave direction, is also widely used in the literature).

Since Equation (3.2.9) is linear and the zero-order terms are constant, the use of Equation (3.2.10) corresponds to a Fourier transformation. In fact, more complicated cases can only be solved by the proper Fourier method. Plugging in Equation (3.2.10) (or Fourier transformation) turns the derivatives into factors,

$$\frac{\partial}{\partial t} \to -i\omega, \ \nabla \to i\mathbf{k} \quad .$$

Equation (3.2.9) becomes

$$(\frac{5}{3} p_0 k^2 - \rho_0 \omega^2) \mathbf{V}_1^z + (\frac{k^2 B_0^2}{4\pi} - \rho_0 \omega^2) \mathbf{V}_1^{\perp} = 0 \quad . \tag{3.2.11}$$

This relation must hold for any wave amplitude $\overline{\mathbf{V}}_1^z$ and $\overline{\mathbf{V}}_1^{\perp}$, which are independent of each other and arbitrary. For a non-trivial solution, either of the two expressions in parentheses and the other amplitude must be equal to zero. The two possibilities correspond to two wave modes studied below in detail.

From the first parentheses in Equation (3.2.11) we get

$$\frac{\omega^2}{k^2} = \frac{5 p_0}{3 \rho_0} =: c_s^2 \quad . \tag{3.2.12}$$

A connection between $\omega$ and $k$ as in Equation (3.2.12) is generally called a *dispersion relation*. The wave is named *sound wave* and the phase velocity of this wave, $\omega/k = c_s$, is the sound velocity. The restoring force, as indicated by the numerator, is a gradient in pressure. The factor 5/3 is the ratio of specific heats corresponding to the 3 degrees of freedom of each particle in a plasma. In a completely ionized plasma with solar abundances $c_s = 1.51 \cdot 10^4 \sqrt{T}$ [cm s$^{-1}$], where $T$ is in degrees kelvin. The group velocity of the wave, being the speed at which the wave energy is carried, equals $\partial\omega/\partial k$ and in this case is also $c_s$.

The second expression in parentheses in Equation (3.2.11) describes a completely different wave,

$$\frac{\omega^2}{k^2} = \frac{B_0^2}{4\pi\rho_0} =: c_A^2 \quad . \tag{3.2.13}$$

The wave exists only if $\mathbf{B_0} \neq 0$. It is the *Alfvén wave*, named after its discoverer who received the Nobel Prize in 1969 for this and other contributions. (It is also called the shear Alfvén wave to distinguish it from the compressional Alfvén wave introduced later.) The phase velocity of the waves, $c_A$, is called the Alfvén velocity. We shall use the relation $c_A \approx c(m_e/m_p)^{1/2}\Omega_e/\omega_p^e$ for a hydrogen plasma, and, for solar abundances, its numerical value $2.03 \cdot 10^{11} B_0/\sqrt{n_e}$ [cm s$^{-1}$].

What is the difference between the two wave modes? Let us investigate their polarization (meaning the directions in which the sinusoidal disturbances oscillate). The sound wave has an amplitude $\mathbf{V}_1^z$ and the particles oscillate only in the $z$-direction,

$$\mathbf{V}_1 = \overline{V}_1^z \mathbf{e}_z e^{i(kz-\omega t)} \quad , \tag{3.2.14}$$

where $\mathbf{e}_z$ is the unit vector in the $z$-direction. Such a wave in which the particles oscillate in the direction of propagation is called *longitudinal*. Since $\mathbf{V}_1$ is parallel to $\mathbf{B_0}$, it follows from Equation (3.2.6) that

$$-i\omega\mathbf{B}_1 = ik\mathbf{e}_z \times (\mathbf{V}_1 \times \mathbf{B}_0) = 0 \quad . \tag{3.2.15}$$

Faraday's equation and Equation (3.2.15) also require that $\mathbf{E}_1 = 0$. Equations (3.2.4) and (3.2.7) give

$$\rho_1 = \rho_0 \frac{k}{\omega} V_1^z \tag{3.2.16}$$

$$p_1 = \frac{5}{3} p_0 \frac{k}{\omega} V_1^z \quad . \tag{3.2.17}$$

Thus a sound wave parallel to $\mathbf{B_0}$ does not cause any electric or magnetic disturbances in a plasma, but oscillates in density and pressure like in a neutral gas. These waves are purely hydrodynamic.

On the other hand, the particles in Alfvén waves oscillate in transverse motion to the magnetic field and the direction of propagation, since

$$\mathbf{V}_1 = \overline{\mathbf{V}}_1^{\perp} e^{i(kz-\omega t)} \quad . \tag{3.2.18}$$

The magnetic disturbance is perpendicular to $\mathbf{B}_0$ and 180° out of phase with $\mathbf{V}_1$, as it follows from Equation (3.2.6) that



$$\mathbf{B}_1 = -\frac{1}{\omega}\mathbf{k} \times (\mathbf{V}_1 \times \mathbf{B}_0) = -\frac{\text{sign}(k)kB_0}{\omega}\mathbf{V}_1 \quad , \tag{3.2.19}$$

$$\mathbf{J}_1 = \frac{c}{4\pi} ik\mathbf{e}_z \times \mathbf{B}_1 \quad . \tag{3.2.20}$$

However, from Equations (3.2.4) and (3.2.7), we find

$$\rho_1 = 0 \ \text{ and } \ p_1 = 0 \quad . \tag{3.2.21}$$

Alfvén waves are therefore purely magnetodynamic, and the perturbations do not compress the plasma. They cause an oscillating ripple on the magnetic field line and may be compared to oscillations of a violin string. As suggested by the expression (3.2.13) of the phase velocity, the restoring force is magnetic tension, $B_0^2/4\pi$, indeed analogous to strings. It is the result of the magnetic gradient term in the equation of motion (3.2.5). Alfvén waves are most unusual in that they are also solutions of the full, non-linear equations, as can easily be shown by substitution. For this reason, the dispersion relation of Alfvén waves does not change for large amplitudes, and the wave does not dissipate energy by non-linear effects. As an important consequence, Alfvén waves can transport energy over long distances, even if the plasma changes gradually, as in a corona.

### 3.2.3. Perpendicular Propagation

Let us now look at the case where the waves propagate in the $x$-direction with $\mathbf{B_0}$ still in the $z$-direction,

$$\mathbf{V}_1 = (\overline{\mathbf{V}}_1^x + \overline{\mathbf{V}}_1^y + \overline{\mathbf{V}}_1^z)e^{i(kx-\omega t)} \quad . \tag{3.2.22}$$

This is representative of all perpendicular waves without loss of generality. Analogous to the parallel case, Equation (3.2.9) gives

$$(-\rho_0\omega^2 + \frac{5}{3}p_0k^2 + \frac{k^2B_0^2}{4\pi})\mathbf{V}_1^x + (-\rho_0\omega^2)\mathbf{V}_1^y + (-\rho_0\omega^2)\mathbf{V}_1^z = 0 \quad . \tag{3.2.23}$$

Again the zeros of the expressions in parentheses are solutions for wave modes. For the second and third expression we find $\omega = 0$; there is no MHD wave perpendicular to $\mathbf{B}_0$ with transverse ($\perp$ $\mathbf{B}$) particle oscillation. We have a mode from the first parenthesis with

$$\frac{\omega^2}{k^2} = \frac{5}{3}\frac{p_0}{\rho_0} + \frac{B_0^2/4\pi}{\rho_0} \ := \ c_s^2 + c_A^2 \quad . \tag{3.2.24}$$

The wave is longitudinal and possesses a combination of acoustic and electromagnetic properties. It is named a *fast magnetoacoustic wave* since it is faster than both sound and Alfvén waves. It is a longitudinal wave. For $c_s \to 0$, the fast magnetoacoustic wave does not behave like the Alfvén wave derived in Equation (3.2.13), although $w/k = c_A$. It is then called the *compressional Alfvén wave*.

### 3.2.4. General Case

Having explored the physics of the waves in the simple cases of parallel and perpendicular propagation, we now look at the intermediate angles to outline briefly how the modes change with angle. The three modes found for parallel and perpendicular directions keep their identity in the general case of skew propagation. This may have been expected from the fact that the disturbances of the three waves are mutually perpendicular in parameter space. However, the physics of the waves changes considerably.

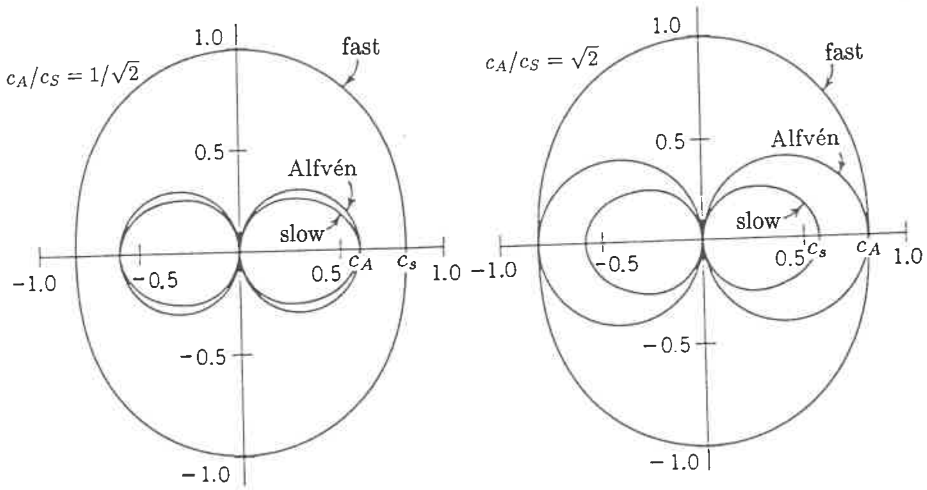


**Fig. 3.1.** Polar diagram of vector phase velocities. The length of the radius vector at an inclination angle $\theta$ to the equilibrium magnetic field equals the phase speed $(\omega/k)$ for waves propagating in that direction. Speeds have been normalized with respect to $(c_A^2 + c_s^2)^{1/2}$.

In non-parallel directions, sound waves are not purely longitudinal, and the Lorentz force takes part in the oscillation through magnetic tension and magnetic pressure. The pure sound wave is a singularity of the parallel and $B = 0$ cases. At a general angle, the mode has a magnetoacoustic nature. However, it does not necessarily coincide with the fast magnetoacoustic mode!

In the parallel direction and for $c_A \geq c_s$, the fast magnetoacoustic wave propagates at the Alfvén velocity and becomes a purely transverse wave driven by magnetic tension. It produces no compression and is physically identical to the Alfvén wave (right part of Figure 3.1). However, for $c_A \leq c_s$ (left part of Figure 3.1) the fast magnetoacoustic wave in the parallel direction is identical to the sound wave. Except for the singularities at parallel and perpendicular propagation, there is always a *fast and a slow magnetoacoustic mode.*

The phase velocity of the Alfvén wave approaches zero at perpendicular phase velocity, since the relevant field component of the magnetic tension decreases with the sine of the inclination angle. Its speed is *intermediate* and always faster than the slow magnetoacoustic mode. The name 'intermediate mode' is not commonly used for linear Alfvén waves, but for non-linear waves (Chapter 9). The slow mode at parallel propagation can be a sound wave or an Alfvén wave, depending on the ratio of $c_A/c_s$. Note that the expressions 'slow', 'intermediate' and 'fast' are mostly name tags of mathematical solutions and do not express the physics of the waves.



Figure 3.1 summarizes the phase velocities of the three MHD modes for different inclinations to the magnetic field and two values of the ratio of Alfvén speed to sound speed. In general, the group velocity, $\partial\omega/\partial k$, has different values and directions from the phase velocity. A remarkable example is formed by the Alfvén (intermediate) waves whose group velocity is always field aligned and thus carries energy only along $\mathbf{B}$ regardless of the inclination of $\mathbf{k}$ .

Some physical properties persist over all angles and $c_A/c_s$ ratios. For fast and slow waves both $\mathbf{V_1}$ and $\mathbf{B_1}$ remain in the plane defined by $\mathbf{B}$ and $\mathbf{k}$. On the other hand, $\mathbf{V_1}$ and $\mathbf{B_1}$ of the Alfvén waves are perpendicular to this plane. For the fast mode, the magnetic pressure and the density always oscillate in phase, but magnetic pressure and density oscillations are 180° out of phase in the slow mode.

## Exercises

**3.1**: Assume a highly ionized hydrogen corona in equilibrium at rest ($V^e = V^p = 0$) with $p_e \approx p_p$. Prove that the gravity of the star creates an electric field

$$eE = \frac{1}{2}(m_p - m_e)g \approx \frac{1}{2}m_p g \tag{3.2.25}$$

in the upward direction. It prohibits the sedimentation of the heavier protons at the bottom of the corona, as one would expect for an atmosphere of neutral particles. Compare the force (3.2.25) to the Lorentz force $F_L = eBv/c$ , where $v/c \approx 1/100$ for a thermal electron in the corona and B $\approx$ 1 G.

**3.2**: Calculate $\beta$ (plasma beta) at the site of a flare before the energy is released, assuming thermal equilibrium and pressure equilibrium, an electron density of $10^{10}$ cm$^{-3}$, a magnetic field of 100 G, and a temperature of $5 \cdot 10^6$ K. Let us first assume for simplicity that the flare locally increases only the electron temperature by a factor of 100 and leaves the other plasma parameters unchanged (neglect non-thermal particles). What is now $\beta$ and what is the consequence? Assume solar abundances, thus $n_i = 0.92\ n_e$. What would happen if only the local magnetic field were annihilated?

**3.3**: Describe the properties of the slow mode MHD waves traveling parallel to $\mathbf{B}$ for both $c_A > c_s$ and $c_A < c_s$.

**3.4**: Prove that in an ideal and non-relativistic MHD plasma the ratio of electric energy density to magnetic energy density is always $(V_\perp/c)^2$, and show that the kinetic and magnetic wave energy are equal for an Alfvén wave.

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